

Noise effects in perfect transmission of quantum states

Fabio Benatti^{a,b1}

Roberto Floreanini^{b2}

Vahid Karimipour^{c3}

^aDipartimento di Fisica, Università degli Studi di Trieste, I-34151 Trieste, Italy,

^bIstituto Nazionale di Fisica Nucleare, Sezione di Trieste, I-34151 Trieste, Italy

^cDepartment of Physics, Sharif University of Technology, P.O. Box 11155-9161, Tehran, Iran.

Abstract

A recent scheme for perfect transmission of quantum states through quasi-one dimensional chains requires application of global control at regular intervals of time. We study the effect of stochastic noise in this control and find that the scheme is robust for reasonable values of disorder. Both un-correlated and correlated noise in the external control are studied and it is remarkably found that the efficiency of the protocol is much higher in presence of correlated noise.

1 Introduction

Since the early developments in the theory of quantum information, the task of coherently transferring quantum states through long and short distance communication lines has been of great importance. While photons are the ideal carriers of quantum information over long distances, it has become evident that the best possible method for transferring quantum information over short distances is to exploit the dynamics of many body systems, specifically of regular arrays of qubits constituting suitable spin chains. In this framework, an arbitrary qubit state is first coupled to the array and then carried to destination by the natural dynamics of the whole system, where it can be extracted with certain fidelity. This idea was first introduced in [1], where it was shown that the natural dynamics of a Heisenberg ferromagnetic spin chain can achieve high-fidelity transfer of qubits over distances as long as 80 lattice units. In contrast to this traditional "passive" protocol, different approaches soon emerged. One idea was to engineer the couplings between the various spins in the chain in such a way that states are transferred with perfect [2]-[9] or with arbitrary high fidelity [10]-[16]; in addition, some minimal external control on the chain dynamics was also introduced in order to achieve similar results [18]-[26].

¹email:benatti@ts.infn.it

²email: roberto.floreanini@ts.infn.it

³email:vahid@.sharif.edu

Quite recently it was shown [27] that particular types of quasi-one dimensional uniformly coupled chains, can achieve perfect state transfer, provided that the natural dynamics of the chain be supplemented with some global control pulses at regular intervals of time. Then it was shown in [27] and using a different geometry in [28], that perfect routing of states in higher dimensional networks, from any point to any other point, is also possible. The advantage of these schemes was that they allowed simultaneous routing of multiple states and also the possibility of overcoming some of the restrictions of the previous protocols, notably introducing some robustness to local imperfections in the network. It is important to note that the introduction of external control is not in contrast with the spirit of quantum state transfer through the natural dynamics of the chain, as long as the external control is global and does not address individual qubits in the network.

While these schemes are to some extent robust against localized imperfections in couplings, i.e. by routing around known defects in the network, the new element of global control inherent in these kinds of schemes brings about the question of their robustness to inaccuracies and imperfections of external control. We can ask to what extent the fidelities of these schemes is affected by imprecision in the timings and the direction of the applied fields in the global pulses, which are necessary for perfect routing of states.

It is on these novel stochastic disturbances that we shall focus in the present work. Specifically we focus on the quasi-one dimensional chain of [27], which is the basic element in higher dimensional uniformly coupled networks. We show that for a moderate value of the noise in global control, one can still achieve a high value of fidelity. Interestingly, we find that the transfer protocol appears to be less affected by correlated noise, as compared to un-correlated one; this result may be of great interest in the actual realization of realistic spin chain channels, since it suggests that externally induced time-correlations may protect the efficiency of the spin transmission lines.

Although in general such noises affect the dynamics of the full chain in an analytically intractable way, we show that the fidelity of the state transfer can be exactly determined and analyzed. More specifically, we are able to compute the fidelity of the protocol, when the direction of the applied field and the timings of the pulses are not precisely tuned because of the presence of external stochastic noise.

The structure of the paper is as follows. In the next section we give a brief account of perfect state transfer in the quasi-one dimensional chain of [27]. Section 3 deals with the general formalism that allows to incorporate in the protocol the effects of the presence of disorder in the pulses. Section 4 is instead devoted to the actual computation of the fidelity both in presence of un-correlated disturbances, and in the more interesting case of correlated noise. The final section contains a brief discussion and outlook.

2 Perfect state transfer in a uniformly coupled quasi-one dimensional chain

The prototype of many-body system that has been used in many protocols is the XY spin chain, consisting of a linear array of N sites, to each of which a spin-1/2 operator with Cartesian components X , Y , Z is attached. The dynamics is then described by a Hamiltonian of the form

$$H = \frac{1}{2} \sum_{m,n} J_{m,n} (X_m X_n + Y_m Y_n) , \quad (1)$$

where the sum is over the various bonds in the array (see Fig.(1) for the specific example discussed in the following). This type of interaction preserves the total component of the spin along the z -direction

$$\left[H, \sum_m Z_m \right] = 0, \quad (2)$$

and moreover does not evolve the uniform background state where all the spin projections along the z direction are up, conventionally called the vacuum state $|\mathbf{0}\rangle$: $H|\mathbf{0}\rangle = 0$. Further, let us denote with $|\mathbf{i}\rangle$ the single excitation state corresponding to the situation in which all spins are up except the one at site i which is down; the collection $\{|\mathbf{i}\rangle\}$, $i = 1, 2, \dots, N$, of these N states, form a basis in the single excitation subspace of the total Hilbert space of the system.

In order to transfer an arbitrary qubit state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ from site 1 to site N , one may apply the following simple protocol. First, embed the qubit into the initial lattice state $\alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle$. Then, let it evolve according to the natural chain dynamics through the lattice until it becomes, at a suitable instant of time, the state $\alpha|\mathbf{0}\rangle + \beta|\mathbf{N}\rangle$; from this state, one can then extract the original qubit from site N .

As mentioned before, one way to achieve this perfect state transfer is to carefully engineer the coupling constants; this has been discussed in [2], where it was shown that a linear XY chain with local couplings of the form $J_{n,m} = \sqrt{n(N-n)}\delta_{n+1,m}$ can indeed perfectly transfer a qubit to the end of the chain, at the specific time $t = \pi$. It is interesting to note that for $N = 2$ and $N = 3$, the couplings will be uniform and indeed it has been shown that these are the only uniformly coupled chains which can achieve perfect state transfer.

The specific chain analyzed in [27] is shown in Fig.(1). The chain is uniformly coupled in the sense that all the couplings have the same modulus. The presence of -1 couplings allows this chain to be broken up into a direct sum of sub-chains with just two and three sites, achieving perfect state transfer. To see this, consider the Hamiltonian pertaining to this chain; with reference to the labeling of the sites of Fig.(1), it can be expressed as

$$H = \left(|\mathbf{1}\rangle\langle\mathbf{2}| + |\mathbf{1}\rangle\langle\mathbf{3}| + h.c. \right) + \left(|\mathbf{2}\rangle\langle\mathbf{4}| - |\mathbf{3}\rangle\langle\mathbf{4}| + h.c. \right) + \left(|\mathbf{4}\rangle\langle\mathbf{5}| + |\mathbf{4}\rangle\langle\mathbf{6}| + h.c. \right) + \dots \quad (3)$$

which can be conveniently re-written as

$$H = \sqrt{2} \left[\left(|\mathbf{1}\rangle\langle(\mathbf{2}, \mathbf{3})_+| + h.c. \right) + \left(|(\mathbf{2}, \mathbf{3})_-\rangle\langle\mathbf{4}| + |\mathbf{4}\rangle\langle(\mathbf{5}, \mathbf{6})_+| + h.c. \right) + \dots \right], \quad (4)$$

where

$$|(\mathbf{k}, \mathbf{k} + \mathbf{1})_{\pm}\rangle = \frac{|\mathbf{k}\rangle \pm |\mathbf{k} + \mathbf{1}\rangle}{\sqrt{2}}.$$

The orthogonality of the states $|(\mathbf{k}, \mathbf{k} + \mathbf{1})_{\pm}\rangle$ shows that the diamond-shaped lattice can be studied as a chain formed by two-site and three-site elementary components as shown in Fig.(1); we will refer to the latter as “virtual chain” and work with it from now on: it consists of $K = (N - 4)/3$ three-site sub-chains plus an initial and a final two-site sub-chain. A convenient basis, in the single-excitation Hilbert sub-space is then given by $\{|\mathbf{i}\rangle\}$, $i = 1, 2, \dots, N$, made of the following orthonormal vectors

$$\begin{aligned} |\mathbf{1}\rangle &\equiv |\mathbf{1}\rangle, & |\mathbf{2}\rangle &\equiv |(\mathbf{2}, \mathbf{3})_+\rangle, & |\mathbf{3}\rangle &\equiv |(\mathbf{2}, \mathbf{3})_-\rangle, & |\mathbf{4}\rangle &\equiv |\mathbf{4}\rangle, \dots \\ & & \dots & & |\mathbf{N} - \mathbf{1}\rangle &\equiv |(\mathbf{N} - \mathbf{1}, \mathbf{N})_-\rangle, & |\mathbf{N}\rangle &\equiv |\mathbf{N}\rangle. \end{aligned} \quad (5)$$

Note that the vector states $|3i + 1\rangle$, $i = 0, 1, \dots, (N - 1)/3$, correspond to single spin down at the site $3i + 1$, like in the diamond-shape representation of the chain; however, unlike in that one, the other vectors represent a single-excitation being in a superposition of a spin down at one site and down at the subsequent one. *From now on we shall work within the representation based on the orthogonal*

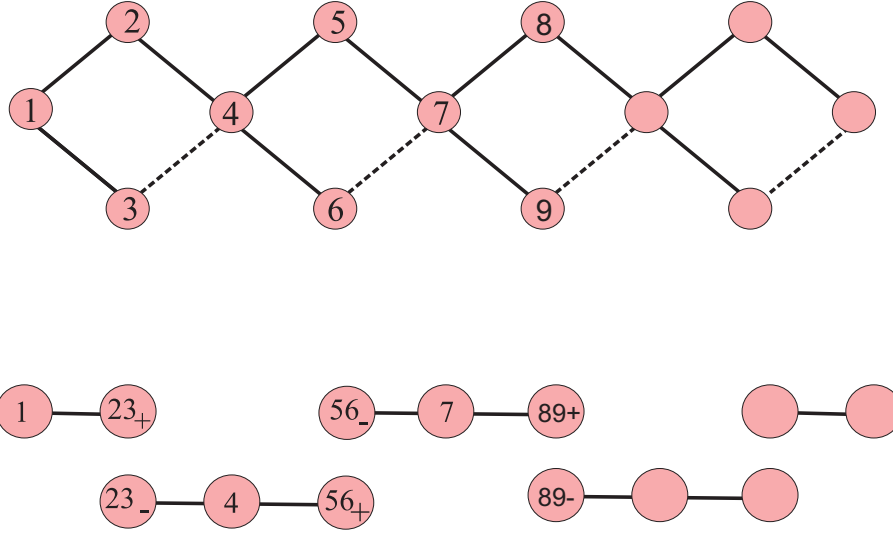


Figure 1: (color online) Above: the quasi-one-dimensional chain introduced in [27]; solid lines indicate bonds with coupling equal to 1, dashed lines those with coupling equal to -1. Below: its equivalent description in terms of two- and three-site perfect state transfer chains.

states non-bold faced states in (5) corresponding to the disjoint union of small chains.

As mentioned before, the time-evolution operator $U_t = \exp(-itH)$ maps the single-excitation sub-space into itself. In the representation corresponding to the above orthonormal basis, it can be split into the following orthogonal sum of blocks

$$U_t = \begin{pmatrix} U_t^{(1)} & & & & \\ & U_t^{(2)} & & & \\ & & U_t^{(3)} & & \\ & & & \ddots & \\ & & & & U_t^{(K+2)} \end{pmatrix}. \quad (6)$$

The unitary blocks $U_t^{(1)}$ and $U_t^{(K+2)}$, corresponding to the initial and final two-site sub-chains, can be represented as

$$U_t^{(1)} = U_t^{(K+2)} = e^{-it\sqrt{2}\sigma_x} = \begin{pmatrix} \cos t\sqrt{2} & -i \sin t\sqrt{2} \\ -i \sin t\sqrt{2} & \cos t\sqrt{2} \end{pmatrix}, \quad (7)$$

by means of the Pauli matrix $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, while the remaining ones can be represented as

$$U_t^{(j)} = e^{-i2t S_x} = \frac{1}{2} \begin{pmatrix} 1 + \cos 2t & -i\sqrt{2} \sin 2t & -1 + \cos 2t \\ -i\sqrt{2} \sin 2t & 2 \cos 2t & -i\sqrt{2} \sin 2t \\ -1 + \cos 2t & -i\sqrt{2} \sin 2t & 1 + \cos 2t \end{pmatrix}, \quad j = 2, 3, \dots, K+1, \quad (8)$$

in terms of the x component of a spin-1 operator:

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (9)$$

The use of σ_x and S_x as generators of the blocks of the unitary time-evolution matrix U_t follows from the expression (4) for the Hamiltonian, whereby σ_x represents the first term in the sum, S_x the second one and so on. One easily sees that, for $t = \pi/(2\sqrt{2})$, respectively $t = \pi/2$, the unitaries in (7), respectively (8), realize perfect state transfers over the elementary two-, respectively three-site sub-chains.

Besides the unitary evolution given by U_t , the transfer protocol introduced in [27] involves instantaneous strong pulses P acting globally on the lower sites of the diamond chain in Fig.(1). The explicit form of the operator P is most simply given in the language of the virtual chain and in the basis (5):

$$P = \begin{pmatrix} 1 & & & & & \\ & \sigma_x & & & & \\ & & 1 & & & \\ & & & \sigma_x & & \\ & & & & \ddots & \\ & & & & & \sigma_x \\ & & & & & & 1 \end{pmatrix}, \quad (10)$$

where σ_x appears $K + 1$ times, each one of them coupling $|i\rangle$ and $|i + 1\rangle$ for $i = 3n + 2$ with $n = 0, 1, \dots, K$. Thus, except for the initial and final sites of the virtual chain, the pulse P transfers any single excitation at an end point of a given sub-chain to that of the contiguous one. The basic idea of the transfer protocol of [27] is to use the unitaries U_t to transfer a single excitation along the sub-chains, while using P to make it jump from one sub-chain to the next. In practice, the composite dynamics of the system, starting from the initial time $t = 0$ to the final time $t = t_f$, is then described by the unitary operator

$$\mathcal{U}_{t_f} = U_{t_f - t_{K+1}} P U_{t_{K+1} - t_K} P \dots U_{t_2 - t_1} P U_{t_1}. \quad (11)$$

Suppose the times t_1, t_2, \dots, t_{K+1} at which the pulses act are chosen as

$$t_1 = \frac{\pi}{2\sqrt{2}}, \quad t_{j+1} = \frac{\pi}{2\sqrt{2}} + j \frac{\pi}{2}, \quad j = 1, 2, \dots, K. \quad (12)$$

Then, from the explicit form that (6), (7) and (8) take in this case, one easily sees that the initial state $\alpha|0\rangle + \beta|1\rangle$, where $|0\rangle$ denotes the ground state, will be perfectly transformed into $\alpha|0\rangle + \beta|N\rangle$ at the final time

$$t_f = \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} K. \quad (13)$$

From this final state, the single qubit $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ can be extracted from site N , thus realizing its perfect transfer from the beginning to the end of the chain.

3 Noise in the protocol

As mentioned in the introduction, in order to transfer a generic qubit state

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \alpha = \cos \frac{\theta}{2}, \quad \beta = \sin \frac{\theta}{2} e^{-i\phi}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad (14)$$

along the chain, one first embeds it into the left hand site of the chain itself as the state

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad (15)$$

where $\{|i\rangle\}_{i=1}^N$ are the basis vectors in (5) and $|0\rangle$ is again the ground state. Under the unitary action of the time-evolution (11), $|\Psi\rangle$ is transformed into $|\Psi_{t_f}\rangle = \mathcal{U}_{t_f}|\Psi\rangle$; since $H|0\rangle = 0$,

$$|\Psi_{t_f}\rangle = \alpha|0\rangle + \beta \left(\sum_{i=1}^N \psi_i(t_f) |i\rangle \right). \quad (16)$$

The protocol purpose is to use the chain dynamics to transfer the state $|\psi\rangle$ from site 1 to site N ; the state of the qubit at site N is obtained by performing the partial trace (tr_N) over single-excitation states involving all sites but the N -th one of the projection $|\Psi_{t_f}\rangle\langle\Psi_{t_f}|$; this gives a 2×2 density matrix

$$\rho_N(t_f) = tr_N |\Psi(t_f)\rangle\langle\Psi(t_f)| = \begin{pmatrix} |\alpha|^2 + |\beta|^2(1 - |\psi_N(t_f)|^2) & \alpha\beta^* \psi_N(t_f)^* \\ \alpha^* \beta \psi_N(t_f) & |\beta|^2 |\psi_N(t_f)|^2 \end{pmatrix}. \quad (17)$$

The robustness of the transmission along the chain can be measured by computing the fidelity of the final mixed state at the site N with respect to the initial pure one embedded at site 1; it is given by

$$F_\psi = \langle\psi|\rho_N(t_f)|\psi\rangle = |\alpha|^2 + 2|\alpha|^2|\beta|^2 \text{Re}(\psi_N(t_f)) + |\beta|^2(|\beta|^2 - |\alpha|^2)|\psi_N(t_f)|^2. \quad (18)$$

Note that it involves only the last coefficient $\psi_N(t_f)$ in the expansion (16). Its average over all input states, given by [1]

$$F := \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta F_\psi = \frac{1}{2} + \frac{1}{6} |\psi_N(t_f)|^2 + \frac{1}{3} \text{Re}(\psi_N(t_f)), \quad (19)$$

is a measure of the overall robustness of the transmission protocol.

The fidelity is maximal, that is $F_\psi = 1$, in the case of the ideal perfect protocol embodied by the unitary evolution \mathcal{U}_{t_f} in (11) with times as in (12) and pulses as in (10). The success of the perfect state transfer depends on the precise control over the timing of the pulses and the direction of their field; in particular, the pulse should be exactly of the form (10) to perfectly transfer the state from one sub-chain to the next. Also the timing of these pulses should be precise and synchronous with the timing required for perfect state transfer within the sub-chains. A disorder in either the field direction or timing of the pulses may drastically reduce the fidelity of the final state.

In the following, we shall consider the case where external noise affects the protocol by modifying precisely these times and pulses. In practice, instead of those in (12), we shall consider modified times of the form (as before, the initial time is set conventionally to zero):

$$\begin{aligned} t_1 &= \frac{\pi}{2\sqrt{2}} + \tau_1, \\ t_{j+1} &= \frac{\pi}{2\sqrt{2}} + j \frac{\pi}{2} + \tau_{j+1}, \quad j = 1, 2, \dots, K, \\ t_f &= \frac{\pi}{\sqrt{2}} + \frac{\pi}{2} K + \tau_{K+2}, \end{aligned} \quad (20)$$

where $\tau = \{\tau_i\}_{i=1}^{K+2}$ is a stochastic process with random variables τ_i distributed according to joint probabilities $\mathcal{P}_{time}(\tau) = \mathcal{P}_{time}(\tau_{K+2}, \tau_{K+1}, \dots, \tau_1)$. The stochastic process may be stationary or not, correlated or not and the stochastic variables may take real values in a discrete or continuous state space. For sake of simplicity, we shall only suppose them to have zero mean-values.

Analogously, we shall consider noisy pulses $P(\theta_i)$ that, while keeping the block form (10), will no longer consist of local pulses represented by σ_x , but by

$$P(\theta_i) := \begin{pmatrix} i \sin \theta_i & \cos \theta_i \\ \cos \theta_i & i \sin \theta_i \end{pmatrix}, \quad (21)$$

that reduces to σ_x in the limit $\theta_i = 0$; here, $\theta = \{\theta_i\}_{i=1}^{K+1}$ is also a stochastic process with probability distribution $\mathcal{P}_{pulse}(\theta) = \mathcal{P}_{pulse}(\theta_{K+1}, \theta_K, \dots, \theta_1)$ consisting of real stochastic variables θ_j with zero mean values.

We shall collect the two stochastic processes into a single one, $\mu = (\tau, \theta)$, with joint probabilities $\mathcal{P}(\mu)$, that may even account for possible correlations between them. Then, in presence of such kind of noise, the unitary evolution (11) will be replaced by

$$\mathcal{U}_{t_f}^{(\mu)} = U(\tau_{K+2} - \tau_{K+1})P(\theta_{K+1})U(\tau_{K+1} - \tau_K) \cdots U(\tau_2 - \tau_1)P(\theta_1)U(\tau_1), \quad (22)$$

where the dependence on the stochastic variables in the various contributions is explicitly shown. Because of its very construction, despite the presence of a perturbing noise, $\mathcal{U}_{t_f}^{(\mu)}$ will map the single-excitation sub-space into itself. Thus, for each realization of the noise, the initial state (15) will be mapped into another single-excitation state

$$|\Psi_{t_f}^{(\mu)}\rangle = \mathcal{U}_{t_f}^{(\mu)}|\Psi\rangle = \alpha|0\rangle + \beta \sum_{i=1}^N \psi_i^{(\mu)}(t_f)|i\rangle, \quad (23)$$

with a reduced density matrix at the N -th site, $\rho_N^{(\mu)}(t_f) = \text{tr}_{\hat{N}}|\Psi^{(\mu)}(t_f)\rangle\langle\Psi^{(\mu)}(t_f)|$, given again by the matrix in (17), but with $\psi_N(t_f)$ replaced by $\psi_N^{(\mu)}(t_f)$, and similarly for the corresponding averaged fidelity $F^{(\mu)}$ (see (19)).

However, a physically meaningful state for the system can only be obtained by averaging over all realizations of the noise; the density matrix representing the state of the chain at the final time t_f is therefore given by

$$\rho(t_f) = \langle\rho^{(\mu)}(t_f)\rangle \equiv \int d\mu \mathcal{P}(\mu) |\Psi_{t_f}^{(\mu)}\rangle\langle\Psi_{t_f}^{(\mu)}|, \quad (24)$$

where the integration region is given by the space spanned by the values that the stochastic variables (τ, θ) can take. Then, the reduced N -site state at time t_f will become

$$\rho_N(t_f) = \text{tr}_N(\rho(t_f)) = \left\langle \text{tr}_N(\rho^{(\mu)}(t_f)) \right\rangle, \quad (25)$$

so that the fidelity averaged over the disorder and over all initial states will be given by

$$\langle F \rangle = \int d\mu \mathcal{P}(\mu) F^{(\mu)} = \frac{1}{2} + \frac{1}{6} \left\langle |\psi_N^{(\mu)}(t_f)|^2 \right\rangle + \frac{1}{3} \left\langle \text{Re}(\psi_N^{(\mu)}(t_f)) \right\rangle. \quad (26)$$

4 Fidelity in the presence of noise

We are now faced with the task of computing the fidelity $\langle F \rangle$ of the state (23) with respect to the initial state to be transferred. For this type of dynamics, it is obviously impossible to determine the explicit form of the final state given any arbitrary initial state. Nevertheless, as explicitly shown in (26), in order to evaluate $\langle F \rangle$, only the coefficient $\psi_N^{(\mu)}(t_f)$ in the expansion (23) is really needed.

As already stressed, the crucial observation is that, in order to determine $\psi_N^{(\mu)}(t_f)$, one needs just follow the change of the last non-zero entry of the vector $|1\rangle$ under the sequential action of operators of the form (7) and (8). In doing so, we shall explicitly write only those components of the transformed vector affected by the various unitary blocks. Let us then consider the initial state $|1\rangle = (1, 0, 0, \dots, 0)^T$; the action of

$$U^{(1)}(\tau_1) = \begin{pmatrix} -\sin(\sqrt{2}\tau_1) & -i\cos(\sqrt{2}\tau_1) \\ -i\cos(\sqrt{2}\tau_1) & -\sin(\sqrt{2}\tau_1) \end{pmatrix}$$

transforms it into $\begin{pmatrix} -\sin(\sqrt{2}\tau_1) \\ -i\cos(\sqrt{2}\tau_1) \end{pmatrix}$. Then, the first pulse $P(\theta_1)$ maps the relevant two-component vector $\begin{pmatrix} -i\cos(\sqrt{2}\tau_1) \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \sin\theta_1\cos(\sqrt{2}\tau_1) \\ -i\cos\theta_1\cos(\sqrt{2}\tau_1) \end{pmatrix}$. After that, the relevant three-component vector $\begin{pmatrix} -i\cos\theta_1\cos(\sqrt{2}\tau_1) \\ 0 \\ 0 \end{pmatrix}$ is turned by $U^{(2)}(\tau_2 - \tau_1)$ into

$$\frac{1}{2} \begin{pmatrix} -i(1 - \cos(2(\tau_2 - \tau_1))) \cos\theta_1 \cos(\sqrt{2}\tau_1) \\ \sqrt{2}\sin(2(\tau_2 - \tau_1)) \cos\theta_1 \cos(\sqrt{2}\tau_1) \\ i(1 + \cos(2(\tau_2 - \tau_1))) \cos\theta_1 \cos(\sqrt{2}\tau_1) \end{pmatrix}. \quad (27)$$

The last component corresponds to the basis vector $|5\rangle$; the second pulse $P(\theta_2)$ turns it into the basis vector $|6\rangle$ multiplying it by $\cos\theta_2$. As such it is then subjected to $U^{(3)}(\tau_3 - \tau_2)$. Continuing in this way, the last non-zero entry of the final vector, that is the coefficient $\psi_N^{(\mu)}(t_f)$ which we need, reads:

$$\begin{aligned} \psi_N^{(\mu)}(t_f) &= (-1)^{K+1} \left[\prod_{i=1}^{K+1} \cos\theta_i \right] \left[\cos(\sqrt{2}\tau_1) \left(\prod_{i=2}^{K+1} \frac{1 + \cos(2(\tau_i - \tau_{i-1}))}{2} \right) \times \right. \\ &\quad \left. \times \cos(\sqrt{2}(\tau_{K+2} - \tau_{K+1})) \right] \equiv (-1)^{K+1} \chi_N^\theta \phi_N^\tau, \end{aligned} \quad (28)$$

where χ_N^θ denotes the first bracket, namely the contribution from random pulses, while ϕ_N^τ that from random time intervals between pulses. By averaging over the noise, one can then compute the mean fidelity $\langle F \rangle$ and thus study the impact of the noise on the robustness of the communication line. In the next subsections we will study in detail state transfer degradation along the chain both in presence of independent and correlated stochastic processes. As we shall see, the transfer protocol turns out to be more robust in the latter case, *i.e.* in presence of time-correlations.

4.1 Independent noise

The most common noise likely to affect spin chain communication lines is that generated by uncorrelated disturbances: it can be described by totally independent stochastic variables $\{\theta_i\}$ and $\{\tau_i\}$ with uniform distributions. In this case, the probability density $\mathcal{P}(\mu) = \mathcal{P}(\theta, \tau)$ factorizes

$$\mathcal{P}(\mu) = \mathcal{P}_{pulse}(\theta) \mathcal{P}_{time}(\tau), \quad (29)$$

$$\mathcal{P}_{pulse}(\theta) = \mathcal{P}_{pulse}(\theta_{K+1}) \mathcal{P}_{pulse}(\theta_K) \dots \mathcal{P}_{pulse}(\theta_1), \quad (30)$$

$$\mathcal{P}_{time}(\tau) = \mathcal{P}_{time}(\tau_{K+2}) \mathcal{P}_{time}(\tau_{K+1}) \dots \mathcal{P}_{time}(\tau_1), \quad (31)$$

where, for simplicity, we have assumed the same probability distribution for all pulses and all time variables. Thus, the computation of $\langle \psi_N^{(\mu)} \rangle$ and $\langle |\psi_N^{(\mu)}|^2 \rangle$, needed in the evaluation of the fidelity, simplifies,

$$\langle \psi_N^{(\mu)} \rangle = (-1)^{K+1} \langle \chi_N^\theta \rangle \langle \phi_N^\tau \rangle, \quad \langle |\psi_N^{(\mu)}|^2 \rangle = \langle (\chi_N^\theta)^2 \rangle \langle (\phi_N^\tau)^2 \rangle, \quad (32)$$

reducing (24) to the product of integrals for each stochastic variable.

In addition, we shall assume the random variables to be uniformly distributed in the intervals $\theta_i \in [-\epsilon_\theta, \epsilon_\theta]$ and $\tau_i \in [-\epsilon_\tau, \epsilon_\tau]$, around the perfect transfer values $\theta_i = 0$ and $\tau_i = 0$, so that $\mathcal{P}_{pulse}(\theta_i) = 1/(2\epsilon_\theta)$ and $\mathcal{P}_{time}(\tau_i) = 1/(2\epsilon_\tau)$. However, note that, besides on τ_1 , the quantity ϕ_N^τ in (28) depends also on differences of the variables τ_i . Therefore, in evaluating the averages $\langle \phi_N^\tau \rangle$ and $\langle (\phi_N^\tau)^2 \rangle$, it is convenient to introduce a new set of independent random variables, $\delta_i \equiv \tau_i - \tau_{i-1}$,

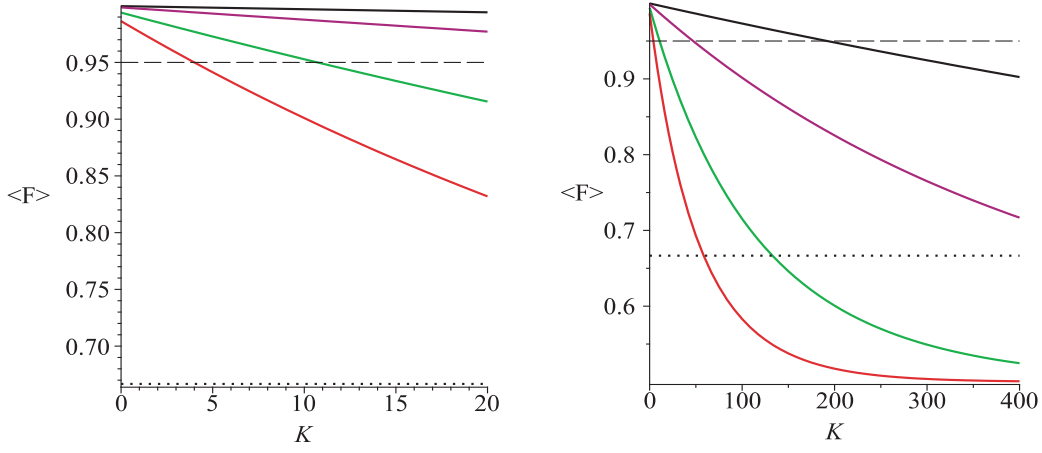


Figure 2: (color online) Average fidelity for various values of $\epsilon_\theta = \epsilon_\tau = \epsilon$ in terms of the chain length. (left figure, for a short chains of length up to ≈ 60 , right figure for long chains of length up to ≈ 1200 : $N \approx 3K$). From top to bottom, $\epsilon = 0.01, 0.02, 0.04$, and 0.06 . The dashed line represents a fidelity of 0.95 and the dotted line represents the maximum fidelity, $F = 2/3$, reached using a classical communication line.

$i = 2, \dots, K + 2$; being linear combinations of two uniformly distributed stochastic variables, these differences obey a triangular distribution [33]:

$$\mathcal{P}_{time}(\delta_i) = \frac{2\epsilon_\tau - |\delta_i|}{4\epsilon_\tau^2}, \quad -2\epsilon_\tau \leq \delta_i \leq 2\epsilon_\tau. \quad (33)$$

Taking this into account, the explicit computation yields:

$$\langle \psi_N^{(\mu)}(t_f) \rangle = (-1)^{K+1} \left[\frac{\sin \epsilon_\theta}{\epsilon_\theta} \right]^{K+1} \left[\frac{\sin(\sqrt{2}\epsilon_\tau)}{\sqrt{2}\epsilon_\tau} \right]^3 \left[\frac{1}{2} \left(1 + \left(\frac{\sin 2\epsilon_\tau}{2\epsilon_\tau} \right)^2 \right) \right]^K, \quad (34)$$

$$\begin{aligned} \langle |\psi_N^{(\mu)}(t_f)|^2 \rangle &= \left[\frac{1}{2} \left(1 + \frac{\sin 2\epsilon_\theta}{2\epsilon_\theta} \right) \right]^{K+1} \left[\frac{1}{2} \left(1 + \frac{\sin 2\sqrt{2}\epsilon_\tau}{2\sqrt{2}\epsilon_\tau} \right) \right] \\ &\times \left[\frac{1}{2} \left(1 + \left(\frac{\sin 2\sqrt{2}\epsilon_\tau}{2\sqrt{2}\epsilon_\tau} \right)^2 \right) \right] \left[\frac{1}{8} \left(3 + 4 \left(\frac{\sin 2\epsilon_\tau}{2\epsilon_\tau} \right)^2 + \left(\frac{\sin 4\epsilon_\tau}{4\epsilon_\tau} \right)^2 \right) \right]^K \end{aligned} \quad (35)$$

All terms appearing in the square brackets above are less or equal to one, becoming smaller and smaller as ϵ_θ and ϵ_τ , measuring the strength of the noise, increase. Consequently, as the chain becomes large, both $\langle \psi_N^{(\mu)}(t_f) \rangle$ and $\langle |\psi_N^{(\mu)}(t_f)|^2 \rangle$ become small. Therefore, in presence of uniformly distributed noise, the fidelity $\langle F \rangle$ in (26) as function of the length K of the transmission line, decreases following a power law, the faster the more ϵ_θ and ϵ_τ differ from zero (see Fig.(2)): the advantage of using a quantum communication line with respect to a classical one becomes then rapidly ineffective. Nevertheless, if the errors induced by the noise are reasonably small (less than one percent), the average fidelity remains above its corresponding classical value $F = 2/3$ for quite long chains ($N > 900$).

4.2 Correlated noise

The presence of correlations in the disorder affecting the spin chain is a concrete possibility in actual experimental realizations of the transmission protocol. Indeed, the presence of correlations among

subsequent pulses is likely to occur in practice due to the inevitable imperfections of the apparatus which produces them, as well as in the time intervals between pulse activations. In such instances, correlations are likely to arise, affecting in various ways the robustness of the spin chain transmission line. We shall concentrate on noise involving the pulses, *i.e.* the stochastic process θ ; the disturbances affecting the time-intervals between pulses, described by the process τ , can be similarly treated and will be briefly discussed at the end of the section.⁴

In presence of correlations among pulses, the probability density $\mathcal{P}_{pulse}(\theta)$ can no longer be written as the product of independent probabilities as in (30). The simplest generalization of (30) involves joint probabilities with one-step correlations based on the conditional probabilities

$$\mathcal{P}_{pulse}(\theta_{i+1}|\theta_i) = q \delta(\theta_{i+1} - \theta_i) + (1 - q) \mathcal{P}_{pulse}(\theta_{i+1}) , \quad (36)$$

specifying the probability that the stochastic variable corresponding to the $i + 1$ -th pulse take the value θ_{i+1} conditioned on the stochastic variable corresponding to the preceding pulse having taken the value θ_i , this being valid for all $i = 1, 2, \dots, K$. The parameter $q \in [0, 1]$ measures the amount of correlations between the stochastic variables θ_{i+1} and θ_i , which is maximal for $q = 1$, while, for vanishing q , θ_{i+1} and θ_i are independent stochastic variables. Then, we shall assume the stochastic process $\theta = \{\theta_i\}$ to be characterized by one-step correlations, namely that its joint probability distributions are of the form

$$\mathcal{P}_{pulse}(\theta_{K+1}, \dots, \theta_2, \theta_1) = \mathcal{P}_{pulse}(\theta_{K+1}|\theta_K) \dots \mathcal{P}_{pulse}(\theta_2|\theta_1) \mathcal{P}_{pulse}(\theta_1) . \quad (37)$$

In determining the fidelity in (26), one now needs to use this expression in computing the averages over the noise. In practice, since the noise affects only pulses and not the time-intervals between them, the quantity (28) which enters (24) together with its square modulus, reduces to $(-1)^{K+1} \chi_N^\theta$, thus one needs evaluate integrals of the form

$$\mathcal{I} = \int d^{K+1}\theta f(\theta_{K+1}) \mathcal{P}_{pulse}(\theta_{K+1}|\theta_K) \dots f(\theta_2) \mathcal{P}_{pulse}(\theta_2|\theta_1) f(\theta_1) \mathcal{P}_{pulse}(\theta_1) , \quad (38)$$

where $f(\theta)$ is either $\cos \theta$ or $\cos^2 \theta$.

In order to estimate the effects of correlated noise on the behavior of the averaged fidelity, we shall assume that the stochastic variables θ_i take only three possible values, $-\epsilon_\theta$, 0 and ϵ_θ , with a probability distribution given by ($0 \leq p \leq 1/2$):

$$\mathcal{P}_{pulse}(\theta_i) = \begin{cases} p & \theta_i = \pm \epsilon_\theta , \\ 1 - 2p & \theta_i = 0 . \end{cases} \quad (39)$$

Then, the Dirac delta in (36) becomes a Kronecker delta and the integrals of the form (38) reduce to sums: $\int d^{K+1}\theta \rightarrow \sum_{\theta_1 \dots \theta_{K+1}}$ and can be cast in a compact form by adopting a transition matrix formalism. That is, one introduces the orthonormal vectors

$$|-\epsilon_\theta\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad |\epsilon_\theta\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad (40)$$

a probability vector $|\mathcal{P}_{pulse}\rangle$ with components $\langle \theta | \mathcal{P}_{pulse} \rangle = \mathcal{P}_{pulse}(\theta)$, and collect the conditional probabilities into a 3×3 transition matrix with entries $\langle \theta | \mathbb{P}_{pulse} | \theta' \rangle = \mathcal{P}_{pulse}(\theta|\theta')$. Explicitly, using

⁴Also here, for simplicity, the two processes θ and τ are assumed to be independent; correlations between pulses and time intervals are surely possible in principle, but certainly less likely than those between θ 's and τ 's variables themselves.

(36) and (39), one finds:

$$\mathbb{P}_{pulse} = \begin{pmatrix} q + (1-q)p & (1-q)p & (1-q)p \\ (1-q)(1-2p) & q + (1-q)(1-2p) & (1-q)(1-2p) \\ (1-q)p & (1-q)p & q + (1-q)p \end{pmatrix}, \quad (41)$$

$$|\mathcal{P}_{pulse}\rangle = \begin{pmatrix} p \\ 1-2p \\ p \end{pmatrix}. \quad (42)$$

Further, by introducing the diagonal 3×3 matrix

$$\mathbb{F} = \begin{pmatrix} f(\epsilon_\theta) & & \\ & f(0) & \\ & & f(-\epsilon_\theta) \end{pmatrix}, \quad (43)$$

the average (38) can be formally rewritten as the following matrix element:

$$\mathcal{I} = \langle \Theta | (\mathbb{F} \mathbb{P}_{pulse})^K \mathbb{F} | \mathcal{P}_{pulse} \rangle, \quad (44)$$

where the final vector $\langle \Theta |$ is the sum of the three basis vectors, explicitly given by

$$\langle \Theta | = \sum_{\theta=\{\pm\epsilon_\theta, 0\}} \langle \theta | = (1 \ 1 \ 1). \quad (45)$$

By recalling that $f(\epsilon_\theta)$ is either $\cos \epsilon_\theta$ or its square, this result allows evaluating for any fixed K the effects of correlated pulse noise in the fidelity (26) as a function of the parameters q and p (see Fig.(3)). As expected, the efficiency of the qubit transfer through the chain degrades in presence of the correlated noise, but in a less dramatic way if compared with its behaviour in presence of disturbances with no correlations; indeed, almost perfect transfer is achieved for high correlated noise even when the error parameter p , the probability for the stochastic variables θ_i to differ from the perfect transfer value $\theta_i = 0$, is as large as $1/2$.

In particular, for $q = 1$, *i.e.* when the correlations between subsequent pulses are maximal, one finds:

$$\langle F \rangle = 1 - p + \frac{p}{3} \left[2 (\cos \epsilon_\theta)^{K+1} + (\cos \epsilon_\theta)^{2K+2} \right], \quad (46)$$

clearly showing that external stochastic noise containing correlations, hence some sort of correlations, disturbs in a milder way the spin chain transmission protocol. Specifically, for long chains, as K becomes large, the averaged fidelity reaches the asymptotic value

$$\langle F \rangle \sim 1 - p, \quad (47)$$

which can still be close to unity, provided p is sufficiently small.

This result has to be compared with the one obtained in the case $q = 0$, when no correlations are present and all stochastic variables θ_i are independent. Also in this case, the pulse noise contribution to the averaged fidelity can be exactly computed, yielding:

$$\langle F \rangle = \frac{1}{2} + \frac{1}{6} \left[1 - 4p \sin^2(\epsilon_\theta/2) \right]^{K+1} + \frac{1}{3} \left[1 - 2p \sin^2 \epsilon_\theta \right]^{K+1}. \quad (48)$$

Since $0 \leq p \leq 1/2$, the square brackets above are always ≤ 1 , so that as K increases the fidelity rapidly approaches its asymptotic value of $1/2$; this is precisely the behaviour encountered in the previous section while discussing independent noise.

Similar results are obtained when correlations are present in the stochastic variables τ_i , affecting the time intervals between the pulses: the joint probabilities $\mathcal{P}_{time}(\tau_i|\tau_j)$ can be taken as in (36).

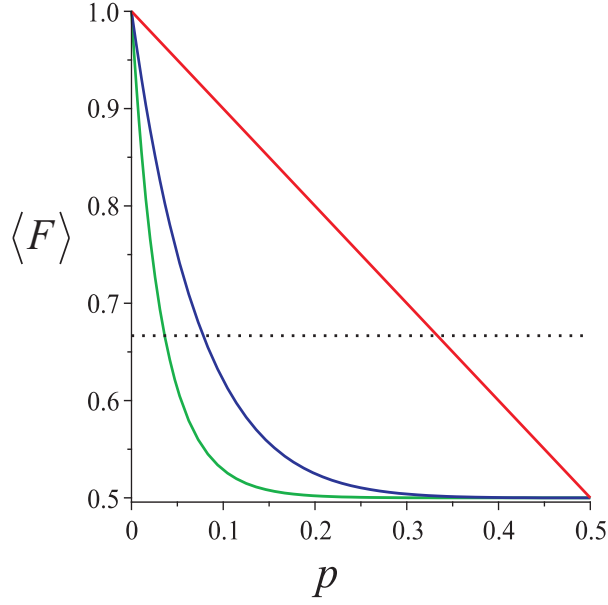


Figure 3: (color online) Average fidelity for various values of q in terms of the error probability p . From bottom to top, $q = 0$ (no correlation), 0.9 and 1 (full correlation). For all the curves $\epsilon = 0.5$ and $K = 100$.

By assuming that the stochastic variables τ_i take only the three possible values $-\epsilon_\tau$, 0 and ϵ_τ , with a probability distribution $\mathcal{P}_{time}(\tau_i)$ similar to the one in (39), the computation of the noise contributions to the averaged fidelity can be treated as in the previous case, leading to contributions of the form (44). However, note that now, except for the first contribution, the matrix \mathbb{F} is no longer diagonal since it involves time differences; indeed, instead of \mathbb{F} in (43), one has to use one with entries: $\langle \tau | \mathbb{F} | \tau' \rangle = f(\tau - \tau')$, where, recalling (28), $f(\epsilon_\tau)$ is either $(1 + \cos 2\epsilon_\tau)/2$, or its square.

The behavior of the averaged fidelity in terms of the probability p , for different values of the correlation parameter q , at fixed K and ϵ_τ , is qualitatively similar to the one discussed before in the case of correlated pulses, given in Fig.(3). In particular, also in this case one observes that the fidelity is less affected by the presence of correlated noise, to the extent that when $q = 1$ it acquires a constant value, independent from the length of the chain:

$$\langle F \rangle = 1 - \frac{2p}{3} \sin^2 2\epsilon_\tau (2 + \sin^2 2\epsilon_\tau) . \quad (49)$$

This result is easily understandable; indeed, when $q = 1$, all intermediate three-site sub-chains (lower picture in Fig.(1)) remain perfect state transfer chains even in presence of noise, and only the first and the last two-site sub-chain fail to transfer the state perfectly, so that the actual length of the transmission line becomes effectively irrelevant.

5 Discussion

We have studied the effect of imperfections in the external control in schemes for perfect transmission of quantum states through a quasi-dimensional chain [27]. Such chains are to be connected to each other to form larger two and three dimensional networks [27],[28] through which qubit states are to be routed from any point to any other point through the natural dynamics of the underlying XY Hamiltonian when assisted by global control pulses. These protocols are by construction robust against known

localized imperfections in the network structure (un-desired couplings, etc). Our study shows that as long as the quasi-one dimensional chains is concerned, such schemes are also robust against imprecision in the sequence of applied global pulses, at least for moderate level of noise and for moderate lengths of the chain. Remarkably we have found that when the noise in successive applications of the pulses are correlated, the efficiency of the protocol is less damaged compared with the case when there is no correlations. Despite the complications of natural dynamics intervened by global external pulses, we have been able to derive exact expressions for the fidelity of state transfer, by taking advantage of the sequential dynamics of the many-body state and following only the evolution of the relevant coefficient which is necessary for calculation of the fidelity.

Acknowledgements:

V. K. would like to thank The Abdus Salam ICTP for its hospitality during the summer 2012, where most of this paper has been prepared.

References

- [1] S. Bose, Phys. Rev. Lett. 91, 207901 (2003).
- [2] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Phys. Rev. Lett. 92, 187902 (2004).
- [3] C. Albanese, M. Christandl, N. Datta, and A. Ekert, Phys. Rev. Lett. 93, 230502 (2004).
- [4] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, and A. J. Landahl, Phys. Rev. A 71, 032312 (2005).
- [5] Man-Hong Yung, and S. Bose, Phys. Rev. A 71, 032310 (2005).
- [6] P. Karbach and J. Stolze, Phys. Rev. A 72, 030301(R) (2005).
- [7] C. Di Franco, M. Paternostro, and M. S. Kim, Phys. Rev. Lett. 101, 230502 (2008).
- [8] G. Gualdi, V. Kostak, I. Marzoli, and P. Tombesi, Phys. Rev. A 78, 022325 (2008).
- [9] M. Markiewicz and M. Wieniak, Phys. Rev. A 79, 054304 (2009).
- [10] T. Shi, Ying Li, Z. Song, and C. Sun, Phys. Rev. A 71, 032309 (2005).
- [11] M.B. Plenio and F.L. Semiao, New J. Phys. 7, 73 (2005).
- [12] A. Wjcek, T. uczak, P. Kurzynski, A. Grudka, T. Gdala, and M. Bednarska, Phys. Rev. A 72, 034303 (2005).
- [13] D. Burgarth, V. Giovannetti and S. Bose, J. Phys. A: Math. Gen. 38 6793 (2005).
- [14] C. Di Franco, M. Paternostro, and M. S. Kim, Phys. Rev. A 81, 022319 (2010).
- [15] E. B. Feldman, E. I. Kuznetsova, and A. I. Zenchuk, Phys. Rev. A 82, 022332 (2010).
- [16] N. Y. Yao, L. Jiang, A. V. Gorshkov, Z.-X. Gong, A. Zhai, L.-M. Duan, and M. D. Lukin, Phys. Rev. Lett. 106, 040505 (2011).
- [17] J. Fitzsimons and J. Twamley, Phys. Rev. Lett. 97, 090502 (2006).
- [18] D. Burgarth, V. Giovannetti, and S. Bose, Physical Review A 75 (6), 062327.

- [19] T. J. Osborne and N. Linden, Phys. Rev. A 69, 052315 (2004).
- [20] D. Burgarth, and S. Bose, Phys. Rev. A 71, 052315 (2005).
- [21] D. Burgarth, V. Giovannetti and S. Bose, Phys. Rev. A 75, 062327 (2007).
- [22] P. Cappellaro, C. Ramanathan, and D. G. Cory, Phys. Rev. Lett. 99, 250506 (2007).
- [23] J. Zhang, M. Ditty, D. Burgarth, C. A. Ryan, C. M. Chandrashekar, M. Laforest, O. Moussa, J. Baugh, and R. Laflamme, Physical Review A 80 (1), 012316 (2009).
- [24] H. Wichterich and S. Bose, Phys. Rev. A 79, 060302(R) (2009).
- [25] X. Wang, A. Bayat, S. G. Schirmer, and S. Bose, Phys. Rev. A 81, 032312 (2010).
- [26] L. Banchi, T. J. G. Apollaro, A. Cuccoli, R. Vaia, and P. Verrucchi, Phys. Rev. A 82, 052321 (2010).
- [27] P. J. Pemberton-Ross and A. Kay, Phys. Rev. Lett. 106, 020503 (2011).
- [28] V. Karimipour, M. Sarmadi Rad and M. Asoudeh, Phys. Rev. A (Rapid Communication), 85, 010302 (2012).
- [29] G. De Chiara, D. Rossini, S. Montangero and R. Fazio, Phys. Rev. A 72, 012323 (2005).
- [30] S. Yang, A. Bayat, and S. Bose, Phys. Rev. A 82, 022336 (2010) .
- [31] A. Bayat, D. Burgarth, S. Mancini, and S. Bose, Phys. Rev. A 77, 050306(R) (2008).
- [32] D.X. Kong and A.M. Wang, Eur. Phys. J. D 55, 211 (2009).
- [33] C.M. Grinstead and J.L. Snell, *Introduction to Probability* (American Mathematical Society, Providence, 1997).